

HANDOUT - GROWTH RATES AND LOGARITHMS

I. What is a natural logarithm (log)?

Let's begin with e . e is a number, approximately equal to 2.7. This number has some special properties and is often used in growth rate calculations. The *exponential function* is simply raising e to an exponent. For example, $\exp(2)$ is e^2 , or the square of e . The *logarithmic function* is the inverse of the exponential function. The log of a number is the exponent x that you would have to raise e to so that e^x is equal to the argument of the function. For example, $\log(10) = 2.3$ because $e^{2.3} = 10$, or equivalently, $\exp(2.3) = 10$.

II. Rules for working with natural logarithms (logs):

1. The log of a product is the sum of the logs.

$$\log(AB) = \log(A) + \log(B)$$

2. The log of an exponential is multiplicative.

$$\log(C^D) = D * \log(C)$$

3. Growth rate rule. A growth rate (as opposed to a percentage change, often used to approximate a growth rate) is defined as the difference between the log of the variable tomorrow and the log of the variable today.

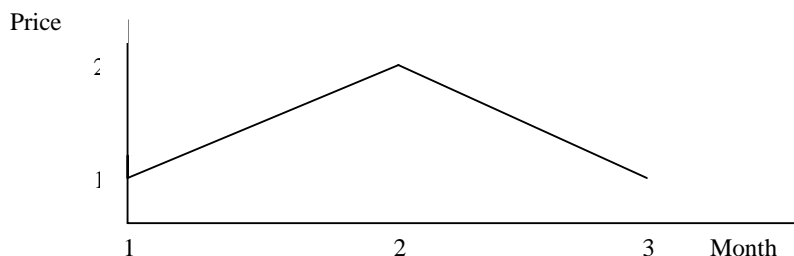
$$g_Y = \log(Y_{t+1}) - \log(Y_t)$$

III. Why use log differences instead of percentage changes?

A log growth rate is a continuous rate of growth. It is a rate of continuous compounding. A percentage change is specific to a particular time horizon. When the time horizons involved are short, a percentage change and a log difference often give very similar answers. However, when we study processes that evolve over many years or decades, like economic growth, the accuracy loss from working with percentage changes can be large.

Another problem that arises in measuring business cycles is that percentage changes don't measure increases and decreases symmetrically. To understand why, just consider a simple example where a price moves from 1 to 2 and back to 1 in consecutive months. Between the first and second months, the price increased $c=$ by 100% $((2-1)/1 * 100)$. But in the following month, the price falls by only 50% $((1-2)/2 * 100)$. If you averaged these two "growth rates" you would come to the conclusion that the average price increase was 25% $((100-50)/2)$. In fact, the price returned back to where it started. There was no price growth in the end at all. If you use percentage changes and then average them over time, you will get answers that are upwardly biased.

One way to think of a log growth rate is that it is the sum of lots of percentage changes, computed over very short time intervals. Consider the price process that switches from 1 to 2 and back to 1. Suppose that price path looked like this:



Now, let's break up time into half-months and compute percentage changes again. This time, we'd get a 50% increase (from 1 to 1.5), then a 33% increase (1.5 to 2), followed by a 25% decrease (2 to 1.5) and a 33% decrease (1.5 to 1). The average growth rate is now only 6.3%. If we computed increases and decreases in $1/4^{\text{th}}$ months, we'd get an average growth rate of 1.6%. As the time increments get finer and finer, we converge to the answer that logs give us, zero change. Logs are limits of percentage changes as the time unit of the percentage change goes to zero.

IV. Deriving the growth rate of output from the production function.

1. Start with a production function, for example, the Cobb-Douglas production function:

$$Y = K^\alpha L^{1-\alpha}.$$

2. Apply the rules of logs numbers 1 and 2 to take the log of both sides of the production function.

$$\log(Y) = \log(K^\alpha L^{1-\alpha}).$$

$$\log(Y) = \alpha \log(K) + (1-\alpha) \log(L)$$

3. Subtract time t equation from $t+1$ equation

$$\log(Y_{t+1}) = \alpha \log(K_{t+1}) + (1-\alpha) \log(L_{t+1})$$

$$-- \log(Y_t) = \alpha \log(K_t) + (1-\alpha) \log(L_t)$$

$$\log(Y_{t+1}) - \log(Y_t) = \alpha (\log(K_{t+1}) - \log(K_t)) + (1-\alpha) (\log(L_{t+1}) - \log(L_t))$$

4. Using the growth rate rule (#3), realize that there are three terms in this equation that can be expressed as growth rates:

$$g_Y = \alpha g_K + (1-\alpha) g_L$$

This equation tells us the relationship between the growth rate of output and the growth rates of capital and labor. Since our growth model tells us about how the capital stock evolves and population projections can give us predictions about labor growth, this expression can be used to forecast output growth.

V. Deriving the growth rate of output per capita.

1. Start with a production function and transform it into a per capita production function.

$$Y = K^\alpha L^{1-\alpha}.$$

Divide both sides by labor, L , to get all quantities in per capita terms.

$$Y/L = (K^\alpha L^{1-\alpha})/L.$$

$$= (K^\alpha L^{1-\alpha})/(L^\alpha L^{1-\alpha})$$

Note that $(L^\alpha L^{1-\alpha}) = L^{\alpha+(1-\alpha)} = L^1 = L$. This is true because multiplying the same variable, each raised to an exponent is the same as raising that variable to the power which is the sum of these exponents.

Rearranging terms,

$$= (K^\alpha/L^\alpha) * (L^{1-\alpha}/L^{1-\alpha})$$

$$= (K/L)^{\alpha} \cdot 1$$

We will use lower case letters to denote quantities in per capita terms:
 $y = Y/L$, $k = K/L$.

$$y = k^{\alpha}$$

This is the Cobb-Douglas production function in per capita terms.

2. Apply rule number 2 to take the log of both sides of the production function.

$$\log(y) = \log(k^{\alpha}).$$

$$\log(y) = \alpha \log(k)$$

3. Subtract time t equation from $t+1$ equation

$$\log(y_{t+1}) = \alpha \log(k_{t+1})$$

$$-- \log(y_t) = \alpha \log(k_t)$$

$$\log(y_{t+1}) - \log(y_t) = \alpha (\log(k_{t+1}) - \log(k_t))$$

4. Using the growth rate rule (#3), realize that there are two terms in this equation that can be expressed as growth rates:

$$g_y = \alpha g_k$$

This tells us that the growth rate of output per capita depends only on the growth rate of the per capita stock of capital.
